On free turbulent convection

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Abstract.

The effect of rising hot blobs in turbulent buoyant convection over a heated horizontal surface is examined. Neglecting the effect of viscosity, it is found that the vertical r.m.s. velocity varies as the square root of height. This result is confirmed by observations in the laboratory as well as in the lower atmosphere. The heat transferred by hot blobs is found to be of the same order of magnitude as the total vertical heat transport. Additionally, when the viscosity is taken into account, a formula is derived for the Nusselt number as a function of the Rayleigh and Prandtl numbers.

1. Introduction.

The intention of this paper is to study turbulent thermal convection over a heated horizontal plane in the absence of a mean velocity. For convection between two horizontal planes a similar motion may occur in a region adjacent to the lower (hotter) plane where the influence of the upper plane is negligible.

A typical feature of turbulent convection is a boundary-like behaviour of the mean temperature profile. Close to the boundary the temperature gradient is very large and conduction is the dominant heat transfer mechanism. At the outer part of the boundary layer, and outside this, heat is transported mainly by convection. It may therefore seem reasonable that in this region the mean temperature gradient, the vertical velocity etc. are independent of the thermal diffusivity κ. Assuming that these quantities also are independent of the kinematic viscosity ν, Priestley (1954), by dimensional considerations, obtained for the mean temperature gradient that

$$- \frac{d\overline{\theta}}{dz} \propto z^{-\frac{4}{3}} \tag{1.1}$$

Here θ is the temperature (potential temperature if the fluid is compressible) and the bar denotes horizontal mean. z is the vertical distance from the lower plane. In the derivation of (1.1) Priestley assumes that the left side depends only upon $H/\rho c_p$, the boyancy parameter g/T_0 and the height z, where H is the (constant) sensible heat flux, ρ is the density, c_p is the specific heat at constant pressure, g is the acceleration of gravity and T_0 a characteristic temperature. The formula (1.1) also follows from the more general theory of Monin and Obukhov (1954) in the limit of zero shear-stress.

In the similarity theory by Priestly $\bar{\theta}$ and the root mean square of the temperature deviation have the same z-dependence. Therefore

$$\tilde{\theta} \propto z^{-\frac{1}{3}} \tag{1.2}$$

where the tilde denotes root mean square deviation from the mean (r.m.s). Since the heat transport in this region is by convection, we may also derive the power law for the vertical velocity, w, from (1.2). Applying that the coefficient of correlation between w and θ is approximately independent of z, which is consistent with the similarity theory, it follows that

$$\tilde{w} \propto z^{\frac{1}{3}} \tag{1.3}$$

This formula was in fact first derived by Prandtl (1932), applying a mixing-length procedure.

The validity of formula (1.1)-(1.3) has been examined in the lower atmosphere as well as in the laboratory. In the case of very weak wind (when the theory is expected to hold) measurements in the atmosphere do not confirm (1.1). For somewhat stronger wind shear, however, the $\frac{1}{3}$-power law seems, surprisingly, to be more closely fulfilled (see for example Deardorff and Willis (1967b)).

Laboratory experiments have been performed among others by Townsend (1959) and Deardorff and Willis (1967a). Townsend measures the temperature variation with height over a heated horizontal surface whereas Deardorff and Willis observe the variation of temperature and velocity between two rigid planes. Townsend concludes that it is not possible to represent his data for the r.m.s. temperature by a $-\frac{1}{3}$-power law and proposes a -0·6-power law as a better approximation for $\tilde{\theta}$. Neither do Deardorff and Willis' diagrams support the power laws obtained from the similarity theory.

It is well known from observations that heat is partly transferred by hot blobs, breaking away from the boundary layer. This process has been emphasized among others by Townsend (1962) and Howard (1966). Presumably these blobs are generated within the boundary layer by some kind of an instability process. A typical feature of these blobs is that their temperatures are nearly conserved.

It is the authors' belief that the blob-mechanism is essential for the heat transfer in free turbulent convection. In the present paper its consequences are studied more closely. We find that this leads to power laws of $\frac{1}{2}$ instead of $\frac{1}{3}$ as in the (1.2) and (1.3) *) The reason why the similarity theory by Priestley does not hold, is that the thermal diffusivity, κ, also plays an important role in the considered region.

2. The blob-mechanism.

Applying the Boussinesq approximation, the vertical momentum equation may be written

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\nabla^2 w + g\alpha\theta \tag{2.1}$$

*) The $\frac{1}{2}$ power law has also been obtained by Long (1973) by a somewhat other line of arguments.

where we have used

$$\rho = \rho_o(1-\alpha\theta) \tag{2.2}$$

as equation of state. Here D/Dt denotes the individual time derivative, ρ_o is the (constant) density outside the boundary layer, p the dynamic pressure, ν is the kinematic viscosity and α the coefficient of expansion. We shall assume that the effect of viscosity may be neglected. We shall also neglect the dynamic pressure term in (2.1) such that the only force is due to boyancy. To neglect the pressure is a plausible approximation in the case of convection over a heated surface. For convection between two rigid planes, however, the presence of the upper plane makes this assumption unvalid in the middle region of the fluid layer.

We shall compare our results mainly with observations from convection between rigid planes. Therefore we take the distance h between the planes as unit of length. Further we choose κ/h, h^2/κ and ΔT as units of velocity, time and temperature, respectively, where ΔT is the temperature difference between the planes. Equation (2.1) may now be written

$$\frac{Dw}{Dt} = Ra\ Pr\ \theta \tag{2.3}$$

where Ra and Pr are the Rayleigh number and the Prandtl number, respectively, defined by

$$Ra = \frac{g\alpha\Delta Th^3}{\kappa\nu}\ , \qquad Pr = \frac{\nu}{\kappa} \tag{2.4}$$

The similarity solution (1.3) is readily obtained from (2.3) by replacing the left side by the characteristic term $w\frac{\partial w}{\partial z}$ and multiplying the equation with w. Interpreting w and θ as the corresponding r.m.s. values, the right side is recognized as the constant heat flux. Hence we obtain $\tilde{w} \propto z^{\frac{1}{3}}$.

Applying (2.3) to a rising hot blob, θ is of the order unity since the blobs are formed in the boundary layer. Utilizing that $w = Dz/Dt$, (2.3) may be written

$$w \frac{Dw}{Dz} = \text{const.} \times \text{Ra Pr} \tag{2.5}$$

where the constant is of order unity . Upon integration

$$w^2 = \text{const.} \times \text{Ra Pr } z \tag{2.6}$$

where z is the vertical distance of travel, which approximately equals the vertical coordinate measured from the bottom plane.

According to our basic hypothesis, the vertical velocity measured at a fixed position in space, is essentially due to the rising (and falling) of blobs. From (2.6) we may then write for the r.m.s. velocity

$$\tilde{w} = A(\text{RaPr})^{\frac{1}{2}} z^{\frac{1}{2}} \tag{2.7}$$

where A is a constant of order unity.

Deardorff and Willis (1967a) have measured the velocity fluctuations in air (Pr = 0,71) for the Rayleigh numbers $6{\cdot}3 \times 10^5$, $2{\cdot}5 \times 10^6$, $1{\cdot}0 \times 10^7$. In figure 1 is shown $\tilde{w}^2$ as function of z obtained from the diagram in the cited paper for the highest Rayleigh number experiment. It is noted that for z-values up to about 0·25 the graph is remarkably close to a straight line, as predicted from (2.7). In this experiment the boundary layer thickness is about 0·1.

From the figure the constant A is found to be $0{\cdot}27$. In figure 2 is displayed $\tilde{w}$ from (2.7) for $Ra = 2{\cdot}5 \times 10^6$ and $Ra = 1{\cdot}0 \times 10^7$. These curves are compared with data taken from Deardorff and Willis' diagrams. We note that the agreement is somewhat poorer for the lowest Rayleigh number. It seems as the Rayleigh number in this case is too small to provide a region where the presence of the upper plane can be neglected. In the same figure is also plotted the trend of the $\frac{1}{3}$- power law obtained from the similarity theory.

According to the observations by Deardorff and Willis (1967a) the coefficient of correlation between θ and w is an approximately constant outside the boundary layer. Since the heat transfer in this region is by convection, $\tilde{\theta}$ is proportional to the inverse power of $\tilde{w}$.

The $-\frac{1}{2}$-power law for the r.m.s. temperature is also supported by the measurements of Myrup (1967) in the lower atmosphere. He finds that $\tilde{\theta}$ varies with height according to the power of $-0{\cdot}47$ in the case of light wind.

It may be of some interest to give a rough estimate of the amount of heat transported with the blobs. The blob-thickness is assumed to be of order δ where δ is the (dimensional) boundary layer thickness. Let L be the characteristic horizontal blob-dimension. Using dimensional quantities, the heat transported by a single blob is then of the order

$$\rho c_p \Delta T \delta L^2 \tag{2.8}$$

In average the heat flux, H, through an area of order unity is given by

$$H = \rho c_p \Delta T \delta \tau^{-1} \tag{2.9}$$

where τ^{-1} is the "break off"-frequency of hot blobs from the boundary layer. The characteristic time τ consists of three intervals; the time period for rebuilding the boundary layer; the time period for the formation of blobs by instability processes and the time for a blob to rise a distance δ. Howard (1966) has proposed that τ is given by the rebuilding time, δ^2/κ, and argues that the formation time usually is smaller. This seems to be confirmed in the experiments by Somerscales and Gazda (1969). It may also be shown that the blob rising time is highest of order δ^2/κ. From the heat equation we readily obtain

$$(\overline{w\theta})_\delta = -\kappa\left(\frac{\partial\overline{\theta}}{\partial z}\right)_0 \tag{2.10}$$

where the bar denotes horizontal mean, and the subscripts 0 and δ refer to the lower plane and outer part of the boundary layer, respectively. Observations indicate that $-\partial\overline{\theta}/\partial z$ is approximately equal to $\Delta T/\delta$. The coefficient of correlation between w and θ is close to $\frac{1}{2}$ outside the boundary layer (Deardorff and Willis 1967a). Therefore, approximately

$$(\tilde{w})_\delta = 2\left(\frac{\Delta T}{\tilde{\theta}}\right)_\delta\left(\frac{\kappa}{\delta}\right) \tag{2.11}$$

The blob rising time $\delta/\tilde{w}$ may then be written

$$\tfrac{1}{2}\left(\frac{\tilde{\theta}}{\Delta T}\right)_\delta\left(\frac{\delta^2}{\kappa}\right) \tag{2.12}$$

According to various observations, $(\tilde{\theta}/\Delta T)_\delta$ is of the order of 10^{-1}. Hence the time interval (2.12) is less (or highest equal) to the rebuilding time for the boundary layer.

Introducing $\tau \sim \delta^2/\kappa$ into (2.9), we then obtain for the heat flux per unit area by blobs

$$H \sim \frac{k\Delta T}{\delta} \tag{2.13}$$

where $k = \rho c_p \kappa$ is the thermal conductivity. The right side is recognized as the heat flux per unit area supplied to the fluid through the lower plane. Thus we conclude that the blob mechanism is sufficiently effective to account for the total heat transport outside the boundary layer.

3. The Nusselt number at very high Rayleigh numbers.

In this section we consider the dimensionless heat flux, the Nusselt number, as a function of the Rayleigh number. For small Prandtl numbers (when the viscosity can be neglected in (2.1)), the Nusselt number can be found from (2.7) and (2.10). Non-dimensionally (2.10) may be written

$$r\,\tilde{w}\,\tilde{\theta} = \beta\,\delta^{-1} \tag{3.1}$$

where the left side is to be evaluated at the outer part of the boundary layer. Further r is the coefficient of correlation between w and θ, and β is a constant close to unity. Introducing $z = \delta$ into (2.7), and utilizing (3.1), we obtain

$$A(PrRa)^{\frac{1}{2}}\delta^{\frac{1}{2}} = \beta(r\,\tilde{\theta}\,\delta)^{-1} \tag{3.2}$$

In accordance with (3.1), the Nusselt number, Nu, is given by

$$Nu = \beta\delta^{-1} \tag{3.3}$$

Hence, from (3.2)

$$Nu = \beta^{\frac{1}{3}}(A\,r\,\tilde{\theta})^{\frac{2}{3}}(PrRa)^{\frac{1}{3}} \tag{3.4}$$

Observations by Deardorff and Willis (1967a) and Somerscales and Gazda (1969) indicate that $\tilde{\theta}$ is not constant, the general trend being a decrease with increasing Prandtl number. The quantities A, β and r, however, are believed to be approximately constant. To make a rough estimate of the coefficient in (3.4), we utilize the data from Deardorff and Willis' measurements in air. Taking $A = 0{\cdot}3$, $\beta = 1{\cdot}2$, $r = 0{\cdot}6$ and $\tilde{\theta} = 0{\cdot}08$, (3.4) yields

$$Nu = 0{\cdot}063(PrRa)^{\frac{1}{3}} \qquad (3.5)$$

This is essentially the same formula as that obtained by Kraichnan (1962) in the limit of small Prandtl numbers ($Pr < 0{\cdot}1$). According to the variation of $\tilde{\theta}$ with Pr, the coefficient in (3.5) is too small. Experiments in mercury ($Pr = 0{\cdot}025$) indicate a coefficient of about $0{\cdot}17$.

For larger Pr the viscosity term in (2.1) can not be neglected. Retaining only characteristic terms in this equation, we may write in non-dimensional form

$$\tilde{w}\tilde{w}_z - Pr\,\tilde{w}_{zz} = PrRa \qquad (3.6)$$

where the subscripts denote partial derivative. Integrating this equation and using boundary layer approximations, we get

$$\tilde{w}^2 + \text{const.x}\,Pr\,\tilde{w}\,\delta^{-1} = \text{const.x}PrRa\delta \qquad (3.7)$$

Inserting from (3.1) into (3.7), and utilizing (3.3), we finally obtain

$$Nu = C_1\tilde{\theta}^{\frac{2}{3}}(1 + C_2\tilde{\theta}Pr)^{-\frac{1}{3}}(PrRa)^{\frac{1}{3}} \qquad (3.8)$$

Here $C_1 = (\beta A^2 r^2)^{\frac{1}{3}}$ and C_2 is a dimensionless constant.

We note from this formula that Nu is proportional to $Ra^{\frac{1}{3}}$ which seems to be the generally accepted relation for turbulent convection. In the limit of small Prandtl numbers it reduces to (3.4). For large Prandtl numbers the formula tends to

$$Nu \propto Ra^{\frac{1}{3}} \qquad (3.9)$$

This form has also been proposed by Kraichnan (1962). As mentioned before, the Prandtl number dependence of $\tilde{\theta}$ is not really known. To make a rough estimate of Nu as function of Ra and Pr, we then have to choose $\tilde{\theta}$ = const. in (3.8). Utilizing once more the data from Deardorff and Willis' observations in air, and taking in addition $C_2\tilde{\theta} = 0{\cdot}15$, equation (3.8) reduces to

$$Nu = 0{\cdot}063(1 + 0{\cdot}15Pr)^{-\frac{1}{3}}(PrRa)^{\frac{1}{3}} \qquad (3.10)$$

In figure 3 we have displayed (3.10) together with data from various experiments.

It is seen that the agreement is relatively good for Prandtl numbers larger than about 0·7. For mercury the value given by (3.10) is markedly too low.

In figure 3 is also displayed the curve based on equation (3.8) with $\tilde{\theta}$ = const. when C_1 and C_2 are chosen to give the "best" overall fit with the observed data. Analytically this curve is given by

$$Nu = 0{\cdot}18(1 + 10Pr)^{-\frac{1}{3}}(PrRa)^{\frac{1}{3}} \qquad (3.11)$$

It seems, however, as if the effect of viscosity is exaggerated in this formula.

In choosing the data points in figure 3 we have required that the Rayleigh number in the original measurements is above 10^6 to secure a fully turbulent motion. Thus in Rossby's experiments we have only considered some of his reported measurements. We also point out that the original measurements presented in figure 3 have some scatter. Somerscales and Gazda, for example, report that the $\frac{1}{3}$-power law covers their data within ±10 per cent.

4. Summary and discussion.

In this paper we have examined the effect of rising hot blobs in turbulent free convection. Neglecting the effect of viscosity, we have found that the vertical r.m.s. velocity varies as the square root of height. This result is confirmed by laboratory measurements in air, and by observations in the lower atmosphere. It is further indicated that the blob mechanism is sufficiently effective to account for the total heat transport outside the boundary layer. A formula giving the Nusselt number as function of the Rayleigh and Prandtl numbers is derived. This formula contains the r.m.s. temperature, $\tilde{\theta}$, which, according to observations, varies somewhat with the Prandtl number. Lacking sufficient information about this variation, we have taken $\tilde{\theta}$ = const. in equation (3.8) in order to compare with experiments.

The present theory is assumed to hold outside the boundary layer in a region where the influence of the upper plane can be neglected. To assure that such a region exists, the boundary layer thickness must be small compared to the distance between the planes. Hence the Nusselt number must be relatively large, above 10, say.

The z-dependence of $\tilde{w}$ (and $\tilde{\theta}$) derived in this paper disagrees with the results obtained from the similarity theory, as put foreward by Priestley (1954). The discrepancy is obviously due to the different roles of κ in the two theories. Priestley neglects the effect of κ in the region outside the boundary layer. In the present theory, however, the heat transport is due to blobs. Since the blobs are formed within the conduction layer, κ clearly must be important in the whole fluid. Also the variation with height of the mean temperature will depend on κ. Disregarding the effect of viscosity, the mean temperature gradient can be written in the form

$$\frac{z}{\Delta T}\frac{d\overline{\theta}}{dz} = f\left(\frac{z}{\delta}\right) \tag{4.1}$$

Here the (dimensional) boundary layer thickness δ is the characteristic length given by

$$\delta = C\frac{\kappa^{\frac{2}{3}}}{(g\alpha\Delta T)^{\frac{1}{3}}} \tag{4.2}$$

where C is a dimensionless constant.

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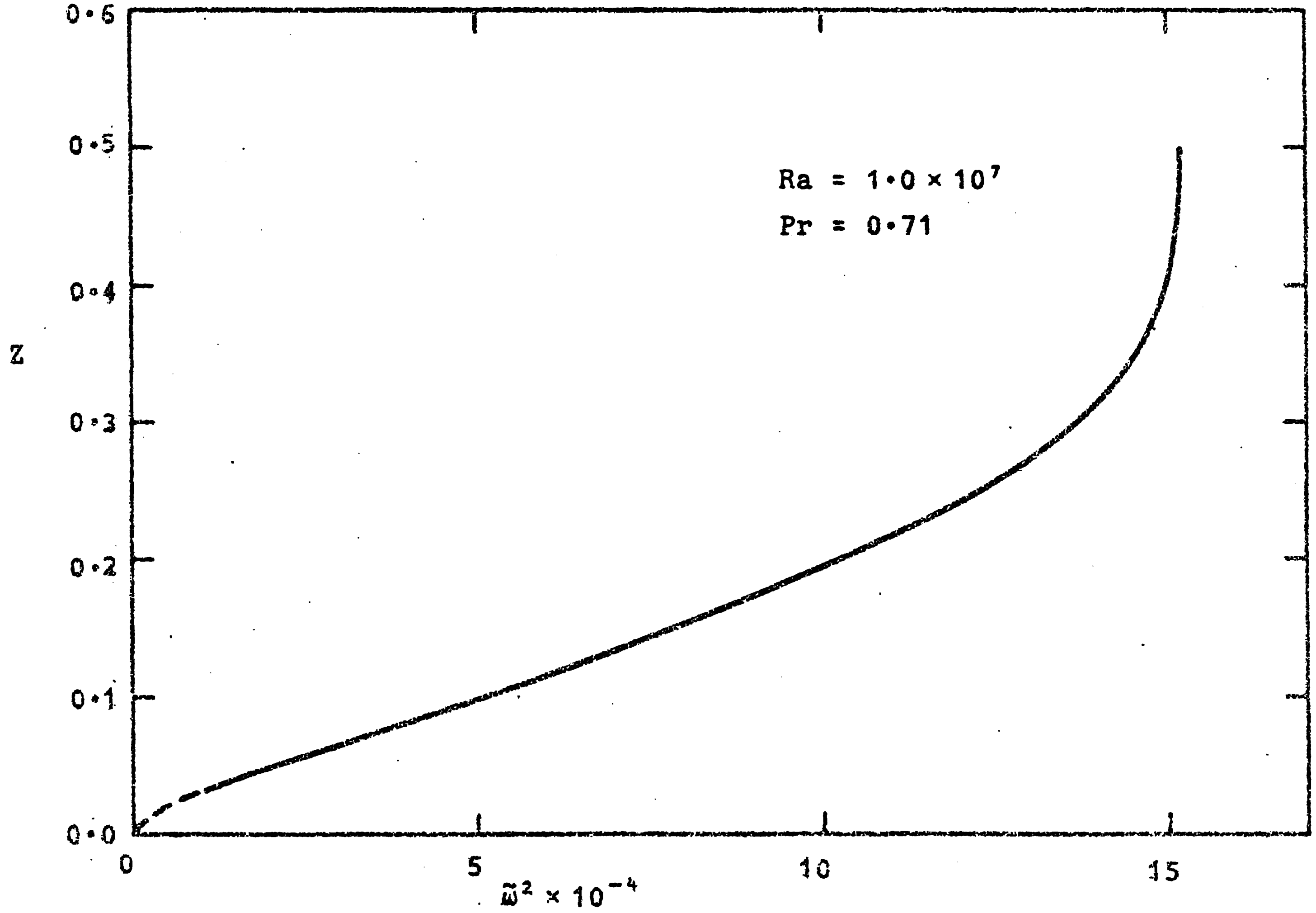
Figure legends

Figure 1. The squared r.m.s. vertical velocity vs height as obtained from Deardorff and Willis' published diagrams.

Figure 2. Comparison of the r.m.s. velocity $\tilde{w}$ (solid curves) from (2.7) with the observations of Deardorff and Willis (1967a) in air (broken curves). —•— , trend of the $\frac{1}{3}$-power law.

Figure 3. Values of $Nu/Ra^{\frac{1}{3}}$ vs Pr obtained from (3.10) (solid curve) and (3.11) (broken curve). Displayed experimental data :

Closed triangle,	Deardorff and Willis (1967a);
-"- circle,	Globe and Dropkin (1959);
-"- square,	Malkus (1954);
Open square,	Mull and Reiher (1930) (taken from Jakob's (1946) interpretation);
-"- triangle,	Rossby (1969);
-"- circle,	Somerscales and Gazda (1969).


0·6
0·5
0·4
0·3
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$\tilde{\omega}^2 \times 10^{-4}$
Ra = 1·0 × 10^7
Pr = 0·71

Figure 1.

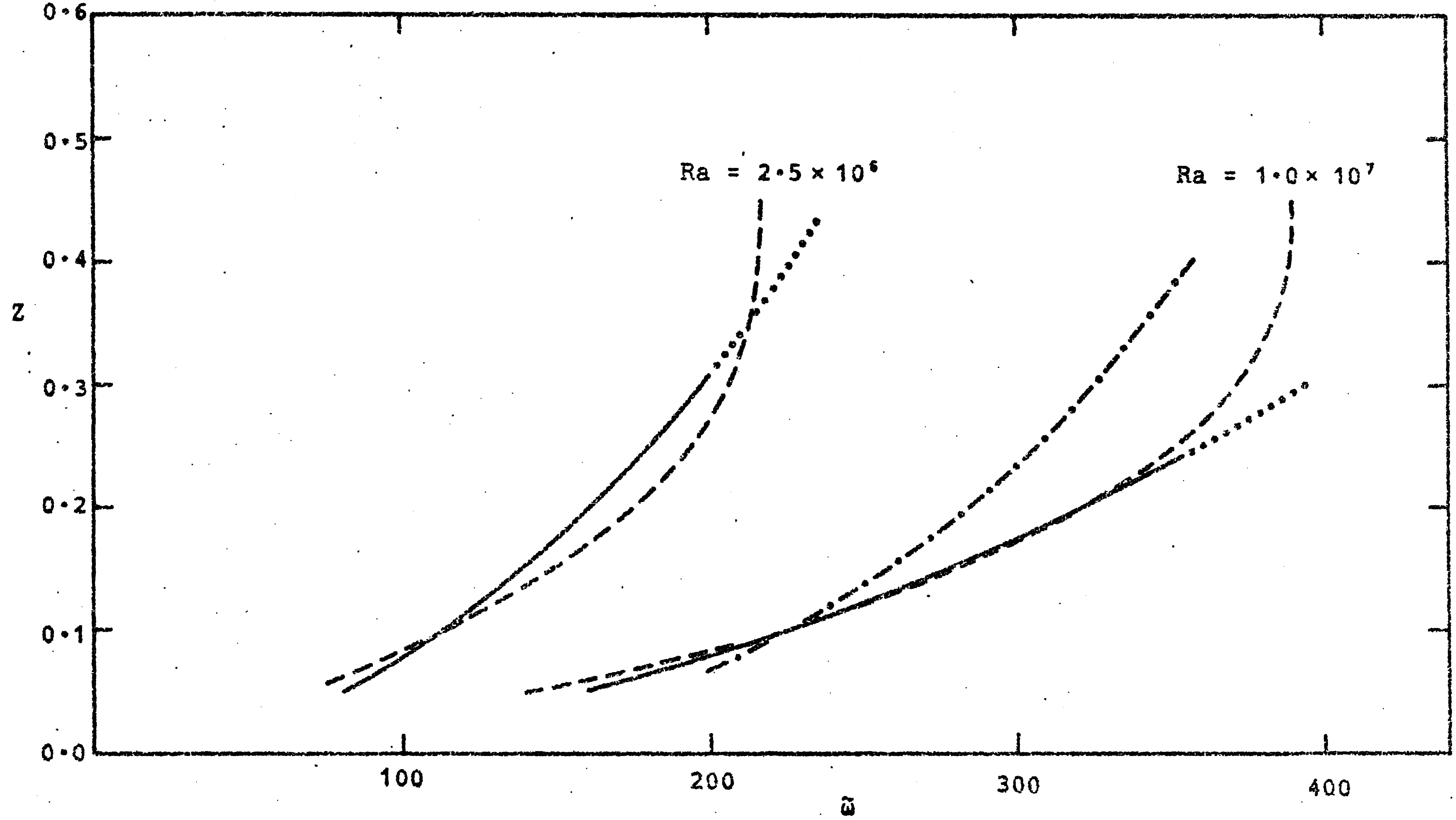

Figure 2.

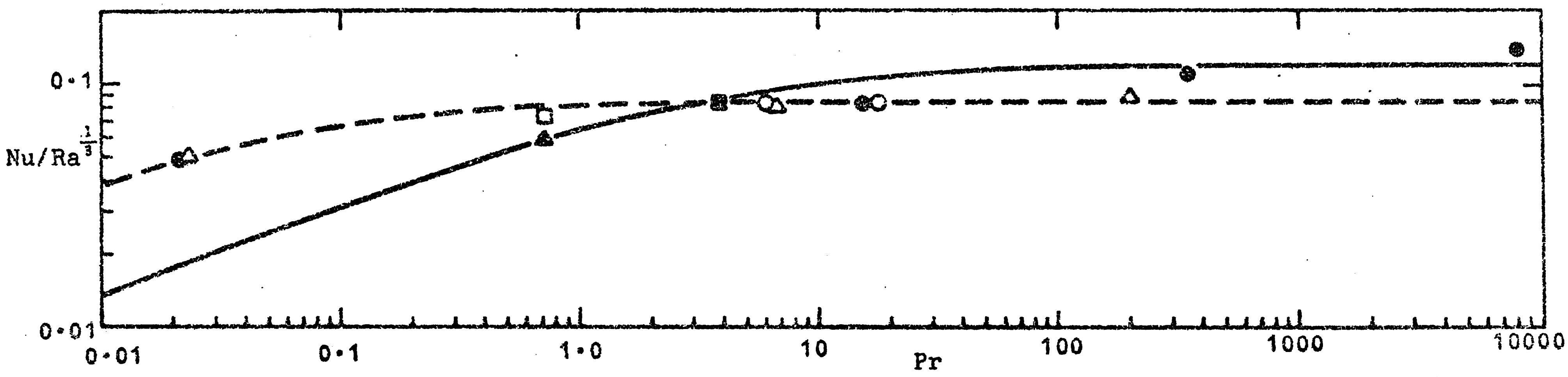

Figure 3.